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ON THE NUMBER OF SUBGRAPHS OF PRESCRIBED TYPE OF PLANAR GRAPHS WITH A GIVEN NUMBER OF VERTICES

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For a planar graph H and a positive integer n we study the maximal number $f = f(n, H)$, such that there exists a planar graph on n vertices containing f subgraphs isomorphic to H. We determine $f(n, H)$ precisely if H is either a complete bipartite (planar) graph or a maximal planar graph without triangles that are not faces, and estimate $f(n, H)$ for every maximal planar graph H.

1. Introduction

All graphs considered are finite, undirected, with no loops and no multiple edges, and unless otherwise specified, they are all planar. For every two graphs, G and H, $N(G, H)$ is the number of subgraphs of G isomorphic to H. G^n is a graph on n vertices. A *triangulation* G^n is a maximal planar graph on n vertices, i.e. a planar graph all of whose faces (including the unbounded face) are triangles. For every graph H and for every $n \geq 1$ put $f(n, H) = \max N(G^n, H)$, where the maximum is taken over all planar graphs G^n. (Clearly, this maximum is attained for some triangulation G^n.) Hakimi and Schmeichel [2] investigated $f(n, H)$, where $H = C_k$ is a cycle of length k. They found that:

$$f(n, C_3) = 3n - 8, \qquad \text{for } n \geq 3, \tag{1}$$

$$f(n, C_4) = (n^2 + 3n - 22)/2, \quad \text{for } n \geq 4, \tag{2}$$

and that for $k \geq 5$ there exist positive constants $c_1(k)$ and $c_2(k)$ such that:

$$c_1(k) \cdot n^{\lfloor k/2 \rfloor} \leq f(n, C_k) \leq c_2(k) \cdot n^{\lfloor k/2 \rfloor}, \quad \text{for all } n \geq k.$$

Here we determine $f(n, H)$ precisely if H is either a complete bipartite (planar) graph or a triangulation without triangles that are not faces, and estimate $f(n, H)$ for every triangulation H.

2. Notation and definitions

$V(G)$ is the set of vertices of the graph G. $G \simeq H$ denotes that the graphs G and H are isomorphic.

If G is a triangulation, a *cut* of G is a triangle in G that is not a face. G is *cut-free* if it includes no cuts. A subgraph of G that is a cut-free triangulation on more than three vertices is called a *block* of G. $b(G)$ is the set of all blocks of G and $c(G)$ is the set of all cuts of G.

A triangulation G is *stacked* if it is C_3 or if every block of G is isomorphic to K_4. (A stacked triangulation is in fact the graph of a stacked 3-polytope, as defined in [3].)

K_k is the complete graph on k vertices, $K_{1,k}$ is the star with k edges, and $K_{2,k}$ is the complete bipartite graph consisting of k independent vertices with two common nonadjacent neighbours. $I(k)$ is the graph consisting of k independent edges. W_k $(k \geq 3)$ is the wheel obtained by joining a new vertex to the k vertices of the cycle C_k.

3. The complete bipartite (planar) graphs

When one considers the problem of determining $f(n, H)$ for various graphs H, it seems natural to begin with the complete planar graphs $K_3 = C_3$ and K_4. However, as was remarked, Hakimi and Schmeichel determined $f(n, K_3)$ for all $n \geq 3$. We shall determine $f(n, K_4)$ in Remark 3 of Section 4 as a special case of Theorem 6. Therefore we begin here with the complete bipartite graphs. The main results of this section are the following two theorems:

Theorem 1. *For every $k \geq 2$ and $n \geq 4$:*

$$f(n, K_{1,k}) = g(n, k), \tag{3}$$

where

$$g(n, k) = 2 \cdot \binom{n-1}{k} + 2 \cdot \binom{3}{k} + (n-4) \cdot \binom{4}{k}.$$

Theorem 2. *For every $k \geq 2$ and $n \geq 4$:*

$$f(n, K_{2,k}) = h(n, k), \tag{4}$$

where

$$h(n,k) = \begin{cases} \binom{n-2}{k}, & \text{if } k \geq 5 \text{ or if } k = 4 \text{ and } n \neq 6, \\ 3 & \text{if } k = 4 \text{ and } n = 6, \\ \binom{n-2}{3} + 3(n-4), & \text{if } k = 3 \text{ and } n \neq 6, \end{cases}$$

$$h(n,k)=\begin{cases}12 & \text{if } k=3 \text{ and } n=6,\\ \binom{n-2}{2}+4n-14, & \text{if } k=2.\end{cases}$$

We begin with a simple lemma.

Lemma 3. *If u, v and w are the degrees of three different vertices of a planar graph G^n, then*

$$u+v+w\leq 2n+2.$$

Proof. Let $x_1, x_2, \ldots, x_n$ be the vertices of G^n and suppose that u, v and w are the degrees of x_1, x_2 and x_3, respectively. Since G^n contains no $K_{3,3}$, there are at most two vertices x_i with $i\geq 4$ that are adjacent to x_1, x_2 and x_3. Thus, the number of edges that join x_1, x_2 and x_3 to some x_i, $i\geq 4$, is at most $2\cdot 3+(n-5)\cdot 2=2n-4$, and we obtain:

$$u+v+w\leq 6+(2n-4)=2n+2. \qquad \square$$

Proof of Theorem 1. Note that if $d_1, d_2, \ldots, d_n$ are the degrees of the vertices of a graph G^n, then

$$N(G^n, K_{1,k})=\sum_{i=1}^{n}\binom{d_i}{k}, \quad \text{for all } k\geq 2.$$

Therefore $f(n, K_{1,k})$ is just

$$\max\sum_{i=1}^{n}\binom{d_i}{k},$$

where the maximum is taken over all degree sequences of planar graphs on n vertices.

For every $n\geq 3$ let S^n be the graph obtained by joining each of two adjacent vertices to each of the $n-2$ vertices of a path of length $n-3$. (Note that $S^3=K_3$ and $S^4=K_4$.)

As is easily checked, for every $k\geq 2$ and $n\geq 4$

$$f(n, K_{1,k})\geq N(S^n, K_{1,k})=g(n,k). \tag{5}$$

In order to complete the proof we have to show that for every $k\geq 2$ and every graph G^n, where $n\geq 4$,

$$N(G^n, K_{1,k})\leq g(n,k). \tag{6}$$

We prove (6) for every fixed k by induction on n. If $n=4$, (6) is trivial. Assuming it holds for $n-1$, let us prove it for n $(n\geq 5)$. Let G^n be a graph. Clearly, we may assume that G^n is a triangulation. Let $d_1 \le d_2 \le \cdots \le d_n$ be the degrees of the vertices of G^n. Euler's formula implies that $\sum_{i=1}^n d_i = 6n - 12$, and clearly $3 \le d_1 \le d_n \le n-1$. As remarked above

$$N(G^n, K_{1,k}) = \sum_{i=1}^n \binom{d_i}{k}. \tag{7}$$

If $\bar{y} = (y_1, y_2, \ldots, y_n)$ and $\bar{z} = (z_1, \ldots, z_n)$ are nondecreasing sequences of positive integers, and if there exist i and j, $1 \le i < j \le n$, such that $z_i = y_i - 1$, $z_j = y_j + 1$ and $z_l = y_l$, for all $l \ne i, j$, then we say that $\bar{z}$ is obtained from $\bar{y}$ by a simple improvement. It is easily checked that in this case

$$\sum_{i=1}^n \binom{y_i}{k} \le \sum_{i=1}^n \binom{z_i}{k}, \quad \text{for all } k \ge 2, \tag{8}$$

and the inequality is strict iff $y_j \ge k-1$.

Returning to our G^n we consider two possible cases.

Case 1. $d_1 \ge 4$.

In this case:

$$4 \cdot n \le \sum_{i=1}^n d_i = 6n - 12,$$

and thus $n \ge 6$. It is easily checked that the vector of length n $(3, 3, 4, \ldots, 4, n-1, n-1)$ can be obtained from $(d_1, \ldots, d_n)$ by a finite sequence of simple improvements. By (7) and (8) we obtain:

$$N(G^n, K_{1,k}) = \sum_{i=1}^n \binom{d_i}{k} \le 2\binom{3}{k} + (n-4)\cdot\binom{4}{k} + 2\cdot\binom{n-1}{k} = g(n,k),$$

as needed.

Case 2. $d_1 = 3$.

Let x be a vertex of degree 3 in G^n, and let u, v and w be the degrees of its three neighbours, where $3 \le u \le v \le w \le n-1$. The number of copies of $K_{1,k}$ in G^n that contain x is precisely

$$\binom{3}{k} + \binom{u-1}{k-1} + \binom{v-1}{k-1} + \binom{w-1}{k-1}.$$

By Lemma 3 $u + v + w \le 2n + 2$. It is easily checked that there exist u', v', and w', $4 \le u' \le v' \le w' \le n-1$, such that $u \le u'$, $v \le v'$, $w \le w'$ and $u' + v' + w' = 2n + 2$. The vector $(3, n-2, n-2)$ can be obtained from $(u'-1, v'-1, w'-1)$ by a finite number of simple improvements. Thus, the number of copies of $K_{1,k}$ in G^n that contain x is:

$$\binom{3}{k}+\binom{u-1}{k-1}+\binom{v-1}{k-1}+\binom{w-1}{k-1}$$
$$\leq \binom{3}{k}+\binom{u'-1}{k-1}+\binom{v'-1}{k-1}+\binom{w'-1}{k-1}$$
$$\leq \binom{3}{k}+\binom{3}{k-1}+2\cdot\binom{n-2}{k-1}. \tag{9}$$

By the induction hypothesis:

$$N(G^n-x,K_{1,k})\leq g(n-1,k). \tag{10}$$

Combining (9) and (10) with the definition of $g(n,k)$ we obtain:

$$N(G^n,K_{1,k})\leq g(n-1,k)+\binom{3}{k}+\binom{3}{k-1}+2\cdot\binom{n-2}{k-1}=g(n,k).$$

This completes the proof for Case 2 and establishes the theorem. □

Remark 1. Theorem 1 states that for every $k\geq 2$ and $n\geq 4$ and for every graph G^n:

$$N(G^n,K_{1,k})\leq g(n,k), \tag{11}$$

and equality holds if G^n is the graph S^n appearing in the proof of the theorem. One can easily check that the proof implies that for $k=2,3,4$ and $n\geq k+1$ equality holds in (11) iff $G^n=S^n$.

Remark 2. The proof of Theorem 1 implies that if $n\geq 12$ and $d_1\leq\cdots\leq d_n$ are the degrees of the vertices of a triangulation G^n, then:

$$N(G^n,K_{1,2})=\sum_{i=1}^{n}\binom{d_i}{2}\geq 12\cdot\binom{5}{2}+(n-12)\cdot\binom{6}{2},$$

since $(d_1,\ldots,d_n)$ can be obtained by a finite sequence of simple improvements from the vector $\bar{c}=(c_1,\ldots,c_n)$, where

$$c_i=\begin{cases}5, & \text{if } i\leq 12,\\ 6, & \text{if } i>12.\end{cases}$$

Since every triangulation G^n contains $\binom{3n-6}{2}$ pairs of edges, and each such pair is either $K_{1,2}$ or $I(2)$, we conclude that:

$$N(G^n,I(2))\leq\binom{3n-6}{2}-12\binom{5}{2}-(n-12)\cdot\binom{6}{2}=(9n^2-69n+162)/2,$$

with equality iff the degree sequence of G^n is $\bar{c}$. In [1] it is proved that such a triangulation G^n exists whenever $n \geq 12$, except for $n = 13$, and thus we obtain:

$$f(n, I(2)) = (9n^2 - 69n + 162)/2,$$

for all $n \geq 12$, except $n = 13$.

The proof of Theorem 2 is similar to that of Theorem 1, although somewhat more complicated. The result of Hakimi and Schmeichel that appears as equation (2) in this paper proves Theorem 2 for $k = 2$. We prove the theorem here for $k \geq 5$ and give only an outline for $k = 3, 4$, since the proof in these cases is rather lengthy and quite similar.

We need two simple lemmas.

Lemma 4. *Let G^n be a (planar) graph that has a vertex x of degree $n-1$, (i.e. x is adjacent to every other vertex of G^n). If $n \geq 5$, then G^n contains two nonadjacent vertices, each of degree ≤ 3.*

Proof.[1] Note that we may assume that G^n is a triangulation. We prove the lemma by induction on n. For $n = 5$ it is trivial. Assuming it holds for all n', $5 \leq n' < n$, let us prove it for n. Let G^n be a triangulation, and let x be a vertex of G^n of degree $n - 1$. Since G^n is a triangulation, there is a Hamiltonian cycle C in $G^n - x$. If no edge of G is a chord of C, then all vertices of C have degree 3 and the assertion of the lemma follows. Thus, we may assume that there is a diagonal joining the vertices y and z of C. This diagonal splits C into two cycles, C_1 and C_2, with a common edge yz. For $i = 1, 2$ let H_i be the induced subgraph of G^n with vertex set $\{x\} \cup C_i$. We claim that H_1 contains a vertex t of degree ≤ 3 in H_1, $t \neq x, y, z$. Indeed, if $|V(H_1)| = 4$ this is trivial, and if $|V(H_1)| \geq 5$ this follows from the induction hypothesis. Similarly H_2 contains a vertex r of degree ≤ 3 in H_2, $r \neq x, y, z$. However, the degree of t in H_1 equals its degree in G^n and the degree of r in H_2 equals its degree in G^n. Thus, t and r are two nonadjacent vertices of G^n, each of degree ≤ 3 in G^n, which completes the proof. □

Lemma 5. *Let G^n be a triangulation and let $d_1 \leq d_2 \leq \cdots \leq d_n$ be its degree sequence. If $4 \leq d_1 \leq d_n \leq n - 2$, then for every $k \geq 5$:*

$$N(G^n, K_{2,k}) \leq \binom{n-2}{k}.$$

Proof. Since G^n includes no $K_{3,3}$, every $K_{1,k}$ in G^n is included in at most one

[1] **Editorial remark.** The following shorter proof was suggested by a referee. $G^n \setminus \{x\}$ is outerplanar, hence it has two nonadjacent vertices ($n \geq 5$) of valences ≤ 2 in $G^n \setminus \{x\}$; they are nonadjacent vertices in G^n, of valences ≤ 3.

$K_{2,k}$ in G^n. Clearly, every $K_{2,k}$ in G^n includes exactly two copies of $K_{1,k}$, and thus

$$N(G^n, K_{2,k}) \le \tfrac{1}{2} N(G^n, K_{1,k}) = \tfrac{1}{2} \sum_{i=1}^{n} \binom{d_i}{k}. \tag{12}$$

It is easily checked that the vector of length n $(4, 4, \ldots, 4, n-2, n-2)$ can be obtained from the vector $(d_1, d_2, \ldots, d_n)$ by a finite sequence of simple improvements. Therefore (12) implies

$$N(G^n, K_{2,k}) \le \tfrac{1}{2} \sum_{i=1}^{n} \binom{d_i}{k} \le \tfrac{1}{2}\left((n-2)\binom{4}{k} + 2\binom{n-2}{k}\right) = \binom{n-2}{k},$$

as needed. □

Proof of Theorem 2 for $k \ge 5$. As is easily checked, for every $k \ge 5$ and $n \ge 4$:

$$f(n, K_{2,k}) \ge N(S^n, K_{2,k}) = \binom{n-2}{k} = h(n,k).$$

In order to complete the proof we have to show that for every $k \ge 5$ and every triangulation G^n, where $n \ge 4$,

$$N(G^n, K_{2,k}) \le \binom{n-2}{k}. \tag{13}$$

We prove (13) for every fixed k by induction on n. If $n \le k+1$, (13) is trivial. Assuming it holds for $n-1$, let us prove it for n $(n \ge k+2)$. Let G^n be a triangulation, and let $d_1 \le d_2 \le \cdots \le d_n$ be the degrees of its vertices. If $d_1 \ge 4$, then by Lemma 4 $d_n \le n-2$ and by Lemma 5 (13) holds, as needed. Thus, we may assume that $d_1 = 3$. Let t be a vertex of degree 3 in G^n, and let x, y and z be its three neighbours. Let k_1, k_2, and k_3 denote the numbers of common neighbours of x and y, y and z, and z and x, respectively, in $V(G^n)\backslash\{x, y, z, t\}$. Since G^n includes no $K_{3,3}$, it can be easily verified that $k_1 + k_2 + k_3 \le n-2$, and if $k_1 = 0$, then $k_1 + k_2 + k_3 \le n-4$. Clearly, $0 \le k_1, k_2, k_3 \le n-4$ and we may assume that $k_1 \le k_2 \le k_3$. It is easily checked that there exist k_1', k_2', and k_3', $1 \le k_1' \le k_2' \le k_3' \le n-4$, such that $k_i \le k_i'$ for $i = 1, 2, 3$ and $k_1' + k_2' + k_3' = n-2$.

The number of $K_{2,k}$'s in G^n that contain t is clearly at most

$$\sum_{i=1}^{3} \binom{k_i + 1}{k-1} \le \sum_{i=1}^{3} \binom{k_i' + 1}{k-1},$$

and since $(1, 1, n-4)$ can be obtained from (k_1', k_2', k_3') by a finite number of simple improvements, this number is at most

$$2 \cdot \binom{2}{k-1} + \binom{n-3}{k-1} = \binom{n-3}{k-1}. \tag{14}$$

By the induction hypothesis:

$$N(G^n - t, K_{2,k}) \leq \binom{n-3}{k}. \tag{15}$$

Combining (14) and (15) we obtain:

$$N(G^n, K_{2,k}) \leq \binom{n-3}{k-1} + \binom{n-3}{k} = \binom{n-2}{k}.$$

This completes the proof and establishes Theorem 2 for $k \geq 5$. □

An outline of the proof of Theorem 2 for $k = 3, 4$. For $n \leq 7$ one can easily prove the theorem by checking all the possible triangulations G^n. Clearly, for $k = 3, 4$ and $n \geq 8$:

$$f(n, K_{2,k}) \geq N(S^n, K_{2,k}) = h(n, k).$$

Thus, we have to show that for $k = 3, 4$ and for every triangulation G^n, where $n \geq 7$,

$$N(G^n, K_{2,k}) \leq h(n, k). \tag{16}$$

We prove (16) for each of the two possible values of k by induction on n. For $n = 7$, (16) holds. Assuming it holds for $n - 1$, let G^n be a triangulation and let $d_1 \leq d_2 \leq \cdots \leq d_n$ be its degree-sequence. If $d_1 = 3$, we proceed exactly as in the proof for $k \geq 5$. Thus, we may assume that $d_1 \geq 4$. By Lemma 4 $d_n \leq n - 2$. If $d_{n-1} = d_n = n - 2$, we can show that G^n must be the graph obtained by joining each of two nonadjacent vertices to each of the $n - 2$ vertices of the cycle C_{n-2} and, as is easily checked in this case, (16) holds. Therefore we may assume that $d_1 \geq 4$, $d_{n-1} \leq n - 3$ and $d_n \leq n - 2$. It is easily seen that in this case the vector of length n $(4, 4, \ldots, 4, 5, n-3, n-2)$ can be obtained from the vector $(d_1, \ldots, d_n)$ by a finite sequence of simple improvements, and using the same argument as in the proof of Lemma 5 we conclude that for $k = 3, 4$ and for every $n \geq 8$:

$$\begin{aligned} N(G_n, K_{2,k}) &\leq \tfrac{1}{2} N(G^n, K_{1,k}) \\ &= \tfrac{1}{2}\left[(n-3)\binom{4}{k} + \binom{5}{k} + \binom{n-3}{k} + \binom{n-2}{k}\right] \\ &\leq h(n, k), \end{aligned}$$

as needed. □

4. The triangulations

The main results of this section are the following two theorems.

Theorem 6. *If H is a cut-free triangulation on k vertices, $k \geq 4$, then*

$$f(n, H) = [(n-3)/(k-3)], \quad \text{for all } n \geq 3.$$

Theorem 7. *For every triangulation H that contains a cut and for every $n \geq 4$:*

$$f(n, H) \leq 12(n-4)/|\operatorname{Aut} H|,$$

where $|\operatorname{Aut} H|$ is the number of automorphisms of H.

(Considering Theorem 6, one can easily verify that Theorem 7 holds for all triangulations with four exceptions: the graphs of the triangule, the tetrahedron, the octahedron, and the icosahedron.)

In order to prove Theorems 6 and 7 we need a few simple lemmas concerning the blocks and the cuts of a triangulation. Since the contents of these lemmas seem to be well known, we shall leave most of the proofs to the reader.

Lemma 8. *Let $G = G^n$ be a triangulation, $n \geq 4$.*

(i) *If T is a cut of G that splits G into two triangulations, A and B, having T as a common face, then $c(G)$ is the (disjoint) union of $c(A)$, $c(B)$ and $\{T\}$, and $b(G)$ is the (disjoint) union of $b(A)$ and $b(B)$.*

(ii) *Every face of G is contained in a unique block of G and every cut of G is contained in precisely two blocks of G.*

(iii) *Let $cb(G)$ denote the graph whose vertex set is $b(G)$, and $B_1, B_2 \in b(G)$ are joined iff their intersection is a cut of G. Then $cb(G)$ is a tree.*

(iv) $|c(G)| = |b(G)| - 1$.

Proof. Most assertions of part (i) can be easily verified. In showing that $b(G) \subset b(A) \cup b(B)$, use the fact that every block of G is a 3-connected graph. Part (ii) and part (iii) are proved by induction on n, using the assertions of the preceding part(s). Part (iv) follows immediately from (iii). □

Lemma 9. *Let G^n be a triangulation, $n \geq 4$.*

(i) *If G^n has q blocks $H_1^{n_1}, H_2^{n_2}, \ldots, H_q^{n_q}$, then*

$$n - 3 = \sum_{i=1}^{q} (n_i - 3).$$

(ii) *The number of cuts in G^n is at most $n-4$, and equality holds iff G^n is a stacked triangulation.*

Proof. Part (i) is proved by induction on q, using part (i) of Lemma 8. Part (ii) follows easily from part (i), using part (iv) of Lemma 8. (Note that a block has at least four vertices, and a block with four vertices is K_4.) □

Lemma 10. *Let $T_1, T_2, \ldots, T_q$ be q cut-free triangulations, each containing more than three vertices. Then there exists a triangulation G with precisely q blocks $H_1, \ldots, H_q$ such that $H_i = T_i$ for $1 \leq i \leq q$.*

Proof. By induction on q. The case $q = 1$ is trivial. If $q > 1$, and F is a triangulation with $q-1$ blocks $H_1, \ldots, H_{q-1}$, isomorphic to $T_1, \ldots, T_{q-1}$, respectively, then the required triangulation G is obtained by gluing together F and an isomorphic copy of T_q along a common face. □

Lemma 11. *Let $H = H^n$ and $F = F^n$ be two cut-free triangulations, $n \geq 4$. Let x_1, x_2, and x_3 be the vertices of a face of H and let y_1, y_2, and y_3 be the vertices of a face of F. Then there exists at most one isomorphism $g: H \to F$ that satisfies $g(x_i) = y_i$, for $1 \leq i \leq 3$.*

Proof. Let g be such an isomorphism. The edge x_1x_2 is included in precisely two triangles of H, one of them is $x_1x_2x_3$. Let the other triangle be x_1x_2a. Similarly, y_1y_2 is included in precisely two triangles of F, $y_1y_2y_3$ and y_1y_2b. Clearly, g must satisfy $g(a) = b$. By repeated application of this argument one can easily show that $g(c)$ is uniquely determined for all $c \in V(H)$. □

Proof of Theorem 6. Let G^n be a triangulation. By definition, every copy of H in G^n is a block of G^n. By part (i) of Lemma 9:

$$n - 3 \geq N(G^n, H) \cdot (k-3).$$

Therefore

$$f(n, H) \leq [(n-3)/(k-3)].$$

Conversely, put $r = [(n-3)/(k-3)]$. By Lemma 10 there is a triangulation $G = G^{r(k-3)+3}$ with r blocks, each isomorphic to H. Thus:

$$f(n, H) \geq f(r(k-3)+3, H) \geq N(G, H) = r = [(n-3)/(k-3)]. \qquad \square$$

Remark 3. The proof of Theorem 6 implies that if H is a cut-free triangulation on k vertices, $k \geq 4$, and if $k-3$ divides $n-3$, then for every triangulation G^n:

$$N(G^n, H) \leq (n-3)/(k-3),$$

and equality holds iff every block of G^n is isomorphic to H. In particular, for every $n \geq 3$ and for every triangulation G^n:

$$N(G^n, K_4) \leq n-3,$$

and equality holds iff G^n is a stacked triangulation.

Note also that Euler's formula and part (ii) of Lemma 9 imply that for every triangulation G^n, $n \geq 3$:

$$N(G^n, K_3) \leq (2n-4) + (n-4) = 3n-8, \tag{17}$$

and equality holds iff G^n is a stacked triangulation. This is just the result of Hakimi and Schmeichel, quoted in equation (1) of this paper.

Proof of Theorem 7. Let H be a triangulation that contains a cut and let G^n be a triangulation, $n \geq 5$. We must show that

$$N(G^n, H) \leq 12(n-4)/|\text{Aut}\, H|. \tag{18}$$

Let C be a cut of H and let B be a block of H that contains C. Every isomorphism of H into G clearly maps C onto some cut T of G and maps B onto a block of G that includes T. But T is included in precisely two blocks of G, say A_1 and A_2. The number of possible maps of C onto T is six. By repeated application of Lemma 11 it is easily shown that there are at most six isomorphisms of H into G that map C onto T and B onto A_1, and there are at most six isomorphisms of H into G that map C onto T and B onto A_2. Therefore there are at most 12 isomorphisms of H into G that map C onto a given cut T of G. By part (ii) of Lemma 9 the number of cuts in G is at most $n-4$, and thus there are at most $12(n-4)$ isomorphisms of H into G. However, the number of such isomorphisms is exactly

$$N(G^n, H) \cdot |\text{Aut}\, H|,$$

which implies (18). □

Remark 4. Recall that S^5 is the graph obtained from K_5 by deleting an edge. By Theorem 7:

$$N(G^n, S^5) \leq 12(n-4)/|\text{Aut}(S^5)| = n-4, \tag{19}$$

for every triangulation G^n, $n \geq 5$. The proof of Theorem 7 implies that equality holds in (19) iff G^n is a stacked triangulation and thus $f(n, S^5) = n-4$, for all $n \geq 5$. This shows that Theorem 7 is, in a sense, the best possible. However, by a slight modification of the proof of Theorem 7 it is not difficult to obtain a better upper bound for $f(n, H)$, if H is a triangulation that has a cut but is not stacked.

Remark 5. For every fixed graph H, the function $\varphi_H(n) = f(n, H)$ is clearly super-additive, and therefore $f(n, H)/n$ tends to a (finite or infinite) limit as $n \to \infty$. By Theorem 7 this limit is finite for every triangulation H.

We conclude the paper with the following conjecture of M.A. Perles.

Conjecture. For every 3-connected (planar) graph H there is a constant $c(H)$ such that

$$f(n, H) \leq c(H) \cdot n, \quad \text{for all } n.$$

(One should note that if $H \neq K_3$ is planar and not 3-connected, then $f(n, H) \geq c(H) \cdot n^2$ for a suitable positive constant $c(H)$ and for all $n \geq |V(H)|$.)

By Theorem 7 the conjecture holds for every triangulation H. We can prove the conjecture if H is any wheel W_k ($k \geq 3$). It is worth noting that unlike the case of the triangulations, the constant $c(H)$ in the conjecture cannot be chosen independently of H, since it can be easily shown that for every $k \geq 2$ and $m \geq 1$:

$$f(m \cdot (4k+1), W_{3k}) \geq m \cdot \binom{2k}{k}.$$

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